

LOWER BOUND FOR THE REGULARITY INDEX OF FAT POINTS

Phan Van Thien

Abstract

The problem to find an upper bound for the regularity index of fat points has been dealt with by many authors. In this paper we give a lower bound for the regularity index of fat points. It is useful for determining the regularity index.

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1 Introduction

Let P_1, \dots, P_s be distinct points in the projective space $\mathbb{P}^n := \mathbb{P}^n(K)$, K an algebraically closed field. Denote by \wp_1, \dots, \wp_s the prime ideals in the polynomial ring $R := K[X_0, \dots, X_n]$ corresponding to the points P_1, \dots, P_s . Let m_1, \dots, m_s be positive integers. We will denote by Z the zero-scheme defined by the ideal $I := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$ and call Z a set of *fat points* in \mathbb{P}^n .

The homogeneous coordinate ring of Z is R/I . This ring is a one-dimensional Cohen-Macaulay graded ring, $R/I = \bigoplus_{t \geq 0} (R/I)_t$, whose multiplicity is

$$e(R/I) = \sum_{i=1}^s \binom{m_i + n - 1}{n}.$$

For every t , the $(R/I)_t$ is a finite dimensional vector space over K . The function $H_{R/I}(t) := \dim_K (R/I)_t$ strictly increases until it reaches the multiplicity $e(R/I)$, at which it stabilizes. The *regularity index* of Z , denote by $\text{reg}(Z)$, is defined to be the least integer t such that $H_{R/I}(t) = e(R/I)$. It is well known that $\text{reg}(Z) = \text{reg}(R/I)$, the Castelnuovo-Mumford regularity of R/I . Hence we will also denote $\text{reg}(Z)$ by $\text{reg}(R/I)$.

The problem to exactly determine the regularity index $\text{reg}(Z)$ is fairly difficult. So, instead of determining $\text{reg}(Z)$, one tries to find an upper bound for it. The problem to find an upper bound for $\text{reg}(Z)$ has been dealt with by many authors (see [1]-[12]).

In this paper we give a lower bound for the regularity index of fat points. The lower bound and upper bound are useful tools for determining the regularity index.

The algebraic method used in this paper as well as in [4].

2 Preliminaries

From now on, we say a j -plane, i.e. a linear j -space. We identify a hyperplane as the linear form defining it.

We will use the following lemmas which have been proved in [4].

Lemma 2.1. [4, Lemma 1] *Let P_1, \dots, P_r, P be distinct points in \mathbb{P}^n and let \wp be the defining ideal of P . If m_1, \dots, m_r and a are positive integers, $J := \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$, and $I = J \cap \wp^a$, then*

$$\operatorname{reg}(R/I) = \max \{a - 1, \operatorname{reg}(R/J), \operatorname{reg}(R/(J + \wp^a))\}.$$

Lemma 2.2. [4, Lemma 3] *Let P_1, \dots, P_r be distinct points in \mathbb{P}^n and a, m_1, \dots, m_r positive integers. Put $J = \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$ and $\wp = (X_1, \dots, X_n)$. Then*

$$\operatorname{reg}(R/(J + \wp^a)) \leq b$$

if and only if $X_0^{b-i}M \in J + \wp^{i+1}$ for every monomial M of degree i in X_1, \dots, X_n , $i = 0, \dots, a - 1$.

Suppose that we can find t hyperplanes H_1, \dots, H_t avoiding P such that $H_1 \cdots H_t M \in J$ for every monomial M of degree i in X_1, \dots, X_n , $i = 0, \dots, a - 1$. Since we can write $H_j = X_0 + G_j$ for some linear form $G_j \in \wp$ for $j = 1, \dots, t$, we get $X_0^t M \in J + \wp^{i+1}$. Therefore, we have the following lemma:

Lemma 2.3. *Assume that H_1, \dots, H_t are hyperplanes avoiding P such that $H_1 \cdots H_t M \in J$ for every monomial M of degree i in X_1, \dots, X_n , $i = 0, \dots, a - 1$. If*

$$\delta \geq \max \{t + i \mid i = 1, \dots, a - 1\}$$

then

$$\operatorname{reg}(R/(J + \wp^a)) \leq \delta.$$

The following lemma has been proved in [12].

Lemma 2.4. [12, Lemma 3.3] *Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in \mathbb{P}^n , and m_1, \dots, m_s be positive integers. Put $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$. If $Y = \{P_{i_1}, \dots, P_{i_r}\}$ is a subset of X and $J = \wp_{i_1}^{m_{i_1}} \cap \dots \cap \wp_{i_r}^{m_{i_r}}$, then*

$$\operatorname{reg}(R/I) \geq \operatorname{reg}(R/J).$$

3 Lower bound for the regularity index of fat points

Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in \mathbb{P}^n and m_1, \dots, m_s be positive integers. Let n_1, \dots, n_s be non-negative integers with $(n_1, \dots, n_s) \neq (0, \dots, 0)$ and $m_i \geq n_i$ for $i = 1, \dots, s$. Put $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, $N = \wp_1^{n_1} \cap \dots \cap \wp_s^{n_s}$ ($\wp_i^{n_i} = R$ if $n_i = 0$). Then we have

$$e(R/I) \geq e(R/N) \text{ and } H_{R/I}(t) \geq H_{R/N}(t).$$

So, we can not compare $\text{reg}(R/I)$ with $\text{reg}(R/N)$ by definition of the regularity index of fat points. In Proposition 3.2 we will prove that $\text{reg}(R/I) \geq \text{reg}(R/N)$.

The first, we get the following result.

Lemma 3.1. *Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in \mathbb{P}^n and $m_1, \dots, m_s, n_1, \dots, n_s$ be positive integers with $m_i \geq n_i$ for $i = 1, \dots, s$. Put $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$ and $N = \wp_1^{n_1} \cap \dots \cap \wp_s^{n_s}$, then*

$$\text{reg}(R/I) \geq \text{reg}(R/N).$$

Proof. In case $m_i = n_i$ for $i = 1, \dots, s$, we have the equality. In case there exists j such that $m_j > n_j$, we may assume that $m_s > n_s$. Put $I_1 = \wp_1^{m_1} \cap \dots \cap \wp_{s-1}^{m_{s-1}} \cap \wp_s^{m_s-1}$. We will prove $\text{reg}(R/I) \geq \text{reg}(R/I_1)$.

Put $J = \wp_1^{m_1} \cap \dots \cap \wp_{s-1}^{m_{s-1}}$. By Lemma 2.2 we have

$$\begin{aligned} \text{reg}(R/(J + \wp_s^{m_s})) &\leq b \\ \Leftrightarrow X_0^{b-i} M &\in J + \wp_s^{i+1} \text{ for every } M = X_1^{c_1} \dots X_n^{c_n}, c_1 + \dots + c_n = i, i = 1, \dots, m_s - 1 \\ \Rightarrow X_0^{b-i} M &\in J + \wp_s^{i+1} \text{ for every } M = X_1^{c_1} \dots X_n^{c_n}, c_1 + \dots + c_n = i, i = 1, \dots, m_s - 2 \\ \Leftrightarrow \text{reg}(R/(J + \wp_s^{m_s-1})) &\leq b. \end{aligned}$$

This implies $\text{reg}(R/(J + \wp_s^{m_s})) \geq \text{reg}(R/(J + \wp_s^{m_s-1}))$. By Lemma 2.1 we have

$$\begin{aligned} \text{reg}(R/I) &= \max \{m_s - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp_s^{m_s}))\}, \\ \text{reg}(R/I_1) &= \max \{m_s - 2, \text{reg}(R/J), \text{reg}(R/(J + \wp_s^{m_s-1}))\}. \end{aligned}$$

Therefore, we get

$$\text{reg}(R/I) \geq \text{reg}(R/I_1).$$

By induction on m_s we get

$$\text{reg}(R/I) \geq \text{reg}(R/(\wp_1^{m_1} \cap \dots \cap \wp_{s-1}^{m_{s-1}} \cap \wp_s^{n_s})).$$

By induction on number of points we get

$$\text{reg}(R/I) \geq \text{reg}(R/N).$$

□

From the above lemma and Lemma 2.4 we get the following proposition.

Proposition 3.2. *Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in \mathbb{P}^n and m_1, \dots, m_s be positive integers. Let n_1, \dots, n_s be non-negative integers with $(n_1, \dots, n_s) \neq (0, \dots, 0)$ and $m_i \geq n_i$ for $i = 1, \dots, s$. Put $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, $N = \wp_1^{n_1} \cap \dots \cap \wp_s^{n_s}$ ($\wp_i^{n_i} = R$ if $n_i = 0$). We have*

$$\text{reg}(R/I) \geq \text{reg}(R/N).$$

Now by using the above proposition we show a lower bound for the regularity index of fat points. We recall that all rational normal curves in \mathbb{P}^j are isomorphic under a linear change of coordinates. Hence, we may assume that the points are on the rational normal curve in \mathbb{P}^j with the parametric equations

$$X_0 = t^j, X_1 = t^{j-1}u, \dots, X_{j-1} = tu^{j-1}, X_j = u^j.$$

Let Q_1, \dots, Q_r be distinct points in \mathbb{P}^n . If there exist a linear change of coordinates φ of \mathbb{P}^n such that the coordinates of the points $\varphi(Q_1), \dots, \varphi(Q_r)$ satisfy the parametric equations

$$X_0 = t^j, X_1 = t^{j-1}u, \dots, X_{j-1} = tu^{j-1}, X_j = u^j, X_{j+1} = 0, \dots, X_n = 0,$$

then we said that Q_1, \dots, Q_r are in Rnc-j in \mathbb{P}^n . So, without loss of generality, we may say that if Q_1, \dots, Q_r are in Rnc-j in \mathbb{P}^n , then their coordinates satisfy the above parametric equations.

The points lying on a line are Rnc-1. The points lying on a rational normal curve in \mathbb{P}^n are Rnc-n in \mathbb{P}^n .

Let $Z = m_1P_1 + \dots + m_sP_s$ be a set of fat points in \mathbb{P}^n . Then the set

$$\{P_{i_1}, \dots, P_{i_q} \in \{P_1, \dots, P_s\} \mid P_{i_1}, \dots, P_{i_q} \text{ are in Rnc-} j\}$$

is non-empty.

If $\frac{m}{j}$ is a rational number, we denote by $\left[\frac{m}{j}\right]$ it's integer part.

The following theorem shows a lower bound for the regularity index of fat points.

Theorem 3.3. *Let $Z = m_1P_1 + \dots + m_sP_s$ be a set of fat points in \mathbb{P}^n . Then,*

$$\text{reg}(Z) \geq \max\{D_j \mid j = 1, \dots, n\},$$

where

$$D_j = \max \left\{ \left\lceil \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rceil \mid P_{i_1}, \dots, P_{i_q} \text{ are in Rnc-} j \right\}.$$

Proof. Suppose that points P_{i_1}, \dots, P_{i_q} of $\{P_1, \dots, P_n\}$ are Rnc-j in \mathbb{P}^n . We may assume that $m_{i_1} \geq \dots \geq m_{i_q}$ (after relabeling the points, if necessary). Let $\wp_{i_1}, \dots, \wp_{i_q}$ be the homogeneous prime ideals of R corresponding to the points P_{i_1}, \dots, P_{i_q} . Put

$$J = \wp_1^{m_{i_1}} \cap \dots \cap \wp_{i_q}^{m_{i_q}}.$$

By Lemma 2.4 we have

$$\text{reg}(Z) \geq \text{reg}(R/J).$$

We will prove that

$$\text{reg}(R/J) \geq \left\lceil \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rceil.$$

Since the points P_{i_1}, \dots, P_{i_q} are in Rnc-j in \mathbb{P}^n , we may assume that their coordinates satisfying parametric equations:

$$X_0 = t^j, X_1 = t^{j-1}u, \dots, X_{j-1} = tu^{j-1}, X_j = u^j, X_{j+1} = 0, \dots, X_n = 0$$

and the points $P_{i_q} = (1, 0, \dots, 0)$. Then $\wp_{i_q} = (X_1, \dots, X_n)$. Put

$$J_1 = \wp_{i_1}^{m_{i_1}} \cap \dots \cap \wp_{i_{q-1}}^{m_{i_{q-1}}}.$$

The first, we will prove that

$$\text{reg}(R/(J_1 + \wp_{i_q}^{m_{i_q}})) \geq \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor.$$

Put $T = \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor$. Consider the monomial $X_0^{T-m_{i_q}} X_1^{m_{i_q}-1}$. If

$$X_0^{T-m_{i_q}} X_1^{m_{i_q}-1} \in J_1 + \wp_{i_q}^{m_{i_q}},$$

then there exists a form $h \in \wp_{i_q}^{m_{i_q}}$ of degree $T-1$ such that

$$X_0^{T-m_{i_q}} X_1^{m_{i_q}-1} + h \in J_1.$$

Since $X_0^{T-m_{i_q}} X_1^{m_{i_q}-1} \in \wp_{i_q}^{m_{i_q}-1}$ and $h \in \wp_{i_q}^{m_{i_q}} \subset \wp_{i_q}^{m_{i_q}-1}$, we have

$$X_0^{T-m_{i_q}} X_1^{m_{i_q}-1} + h \in J_1 \cap \wp_{i_q}^{m_{i_q}-1} = \wp_{i_1}^{m_{i_1}} \cap \dots \cap \wp_{i_{q-1}}^{m_{i_{q-1}}} \cap \wp_{i_q}^{m_{i_q}-1}.$$

Since $m_{i_1} + \dots + m_{i_q} - 1 > j(T-1)$, by Bezout's theorem we have

$$X_0^{T-m_{i_q}} X_1^{m_{i_q}-1} + h$$

vanishing on the points $(1, \lambda, \dots, \lambda^j, 0, \dots, 0) \in \mathbb{P}^n$, for every λ in the field k . This implies

$$\lambda^{m_{i_q}-1} + h(1, \lambda, \dots, \lambda^j, 0, \dots, 0) = 0$$

for every $\lambda \in k$. Since $h \in \wp_{i_q}^{m_{i_q}} = (X_1, \dots, X_n)^{m_{i_q}}$, we have $h(1, \lambda, \dots, \lambda^j, 0, \dots, 0) = 0$ or $h(1, \lambda, \dots, \lambda^j, 0, \dots, 0) = \lambda^{m_{i_q}} g(\lambda)$, for some non-zero polynomial $g \in k[x]$. Hence, $\lambda^{m_{i_q}-1} = 0$ or $\lambda^{m_{i_q}-1} + \lambda^{m_{i_q}} g(\lambda) = 0$ for every $\lambda \in k$, a contradiction. Thus, we get

$$X_0^{T-m_{i_q}} X_1^{m_{i_q}-1} \notin J_1 + \wp_{i_q}^{m_{i_q}}.$$

By Lemma 2.2 we have

$$\text{reg}(R/(J_1 + \wp_{i_q}^{m_{i_q}})) \geq T.$$

Next, by Lemma 2.1 we get

$$\begin{aligned} \text{reg}(R/J) &= \max\{m_{i_q} - 1, \text{reg}(R/J_1), \text{reg}(R/(J_1 + \wp_{i_q}^{m_{i_q}}))\} \\ &\geq \text{reg}(R/(J_1 + \wp_{i_q}^{m_{i_q}})) \geq T. \end{aligned}$$

The proof of Theorem 3.3 is now completed. \square

4 Application of lower bound

The first, by using the lower bound we can compute the regularity index of fat points whose support on a line. This formula was showed by E.D. Davis and A.V. Geramita [5, Corollary 2.3] by using another method.

Proposition 4.1. *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in \mathbb{P}^n . If P_1, \dots, P_s lie on a line, then*

$$\text{reg}(Z) = m_1 + \cdots + m_s - 1.$$

Proof. We may assume that $m_1 \geq \cdots \geq m_s$. Since P_1, \dots, P_s lie on a line, we have $D_1 = m_1 + \cdots + m_s - 1$. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. Then $\text{reg}(Z) = \text{reg}(R/I)$. We will prove that

$$\text{reg}(R/I) = D_1.$$

By Theorem 3.3 we have

$$\text{reg}(R/I) \geq \max\{D_j \mid j = 1, \dots, n\}.$$

So, it suffices to prove by induction on s that

$$\text{reg}(R/I) \leq D_1.$$

Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$, by the inductive assumption, we get

$$\text{reg}(R/J) \leq m_1 + \cdots + m_{s-1} - 1 \leq D_1. \quad (1)$$

Choose $P_s = (1, 0, \dots, 0)$, then $\wp_s = (X_1, \dots, X_n)$. For $j = 1, \dots, s-1$, since P_1, \dots, P_s lie on a line, there exists hyperplane, say H_j , passing throught P_j and avoiding P_s . This implies

$$H_1^{m_1} \cdots H_{s-1}^{m_{s-1}} \in J.$$

Therefore, for every monomial $M = X_1^{c_1} X_2^{c_2} \cdots X_n^{c_n}$ of degree i , $i = 0, \dots, m_s - 1$, we have

$$H_1^{m_1} \cdots H_{s-1}^{m_{s-1}} M \in J.$$

Since $D_1 \geq \max\{m_1 + \cdots + m_{s-1} + i \mid i = 1, \dots, m_s - 1\}$, by Lemma 2.3 we get

$$\text{reg}(R/(J + \wp_s^{m_s})) \leq D_1. \quad (2)$$

From (1), (2) and Lemma 2.1 we get

$$\text{reg}(R/I) \leq D_1.$$

□

Next, we consider the fat points whose support lie on two separate lines.

Proposition 4.2. *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in \mathbb{P}^n . Assume that there exist two lines l_1 and l_2 , $l_1 \cap l_2 = \emptyset$ such that $P_1, \dots, P_s \in l_1 \cup l_2$. Then*

$$\text{reg}(Z) = D_1,$$

where $D_1 = \max\{\sum_{i=1}^q m_{i_l} - 1 \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a line}\}.$

Proof. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. Then $\text{reg}(Z) = \text{reg}(R/I)$. We will prove that

$$\text{reg}(R/I) = D_1.$$

By Theorem 3.3 we have

$$\text{reg}(R/I) \geq D_1.$$

So, it suffices to prove that

$$\text{reg}(R/I) \leq D_1.$$

We may assume that $P_1, \dots, P_r \in l_1$ and $P_{r+1}, \dots, P_s \in l_2$. Choose $P_s = (1, 0, \dots, 0)$, then $\wp_s = (X_1, \dots, X_n)$. Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$ and $a = \max\{m_j | j = 1, \dots, r\}$. We consider two following cases.

Case 1: $r = s - 1$. Then $P_1, \dots, P_{s-1} \in l_1$. Let H be the hyperplane containing l_1 and avoiding P_s . For every monomial M of degree i in X_1, \dots, X_n , $i = 1, \dots, m_s - 1$, we have

$$H^a M \in J.$$

Since $D_1 \geq a + m_s - 1 \geq \max\{a + i | i = 1, \dots, m_s - 1\}$, by Lemma 2.3 we get

$$\text{reg}(R/(J + \wp_s^{m_s})) \leq D_1.$$

Since P_1, \dots, P_{s-1} lie on a line, we get

$$\text{reg}(R/J) = m_1 + \cdots + m_{s-1} - 1 \leq D_1.$$

Therefore, by Lemma 2.1

$$\text{reg}(R/I) = \max\{m_s - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp_s^{m_s}))\} \leq D_1.$$

Case 2: $r \leq s - 2$. We may argue by induction on s . Put $b = \max\{a, m_{r+1} + \cdots + m_{s-1}\}$. Let H_j be the hyperplane containing P_{r+j} , l_1 and avoiding P_s , $j = 1, \dots, s - r - 1$. For every monomial M of degree i in X_1, \dots, X_n , $i = 1, \dots, m_s - 1$, we have

$$H_1^{b-(m_{r+1}+\cdots+m_{s-1})} H_1^{m_{r+1}} \cdots H_{s-r-1}^{m_{s-1}} M \in J,$$

($H_1^0 = R$). Since $D_1 \geq \max\{b + i | i = 1, \dots, m_s - 1\}$, by Lemma 2.3 we get

$$\text{reg}(R/(J + \wp_s^{m_s})) \leq D_1.$$

By the inductive assumption, we get

$$\text{reg}(R/J) \leq D_1.$$

Therefore, by Lemma 2.1

$$\text{reg}(R/I) = \max\{m_s - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp_s^{m_s}))\} \leq D_1.$$

□

Now we consider a set of fat points whose support is in Rnc-j.

Lemma 4.3. *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in \mathbb{P}^n . Suppose that j is the least integer such that P_1, \dots, P_s are in Rnc-j. If t is an integer such that*

$$t \geq \max \left\{ m_l, \left\lceil \frac{m_1 + \cdots + m_{s-1} + j - 1}{j} \right\rceil \mid l = 1, \dots, s-1 \right\},$$

then we can find t hyperplanes, say H_1, \dots, H_t , avoiding P_s such that

$$H_1 \cdots H_t \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

Proof. Since the points P_1, \dots, P_s are in Rnc-j in \mathbb{P}^n , we may assume that their coordinators satisfying parametric equations:

$$X_0 = v^j, X_1 = v^{j-1}u, \dots, X_{j-1} = vu^{j-1}, X_t = u^j, X_{j+1} = 0, \dots, X_n = 0.$$

If $j = 1$, then P_1, \dots, P_s lie on a line. For $j = 1, \dots, s-1$, there exists a hyperplane, say H_j , passing through P_j and avoiding P_s . Then we have $t = m_1 + \cdots + m_{s-1}$ hyperplanes $\underbrace{H_1, \dots, H_1}_{m_1}, \dots, \underbrace{H_{s-1}, \dots, H_{s-1}}_{m_{s-1}}$ avoiding P_s such that

$$H_1^{m_1} \cdots H_{s-1}^{m_{s-1}} \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

If $t \geq 2$, then no $l+2$ points of $\{P_1, \dots, P_s\}$ are on a l -plane for $l < j$. This implies that there does not exist any $(j-1)$ -plane containing $j+1$ points of $\{P_1, \dots, P_s\}$. We will prove the lemma by induction on $\sum_{i=1}^{s-1} m_i$.

We may assume that $m_1 \geq \cdots \geq m_{s-1}$. Since j is the least integer such that P_1, \dots, P_s are in Rnc-j, we have $j \leq s-1$. Let σ_1 be the $(j-1)$ -plane passing through P_1, \dots, P_j . Then σ_1 avoids P_s . Therefore, there is a hyperplane, say L_1 , containing σ_1 and avoiding P_s .

Case $s-1 = j$: Then

$$L_1^t \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}} \subset \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

Case $s-1 \geq j+1$: Since $t \geq \left\lceil \frac{m_1 + \cdots + m_{s-1} + j - 1}{j} \right\rceil$ and $m_1 \geq \cdots \geq m_{s-1}$, we have

$$\begin{aligned} t-1 &\geq \left\lceil \frac{m_1 + \cdots + m_{s-1} + j - 1}{j} \right\rceil - 1 \geq \left\lceil \frac{(j+1)m_{j+1} - 1}{j} \right\rceil \\ &\geq m_{j+1}. \end{aligned}$$

On the other hand, since $t \geq \left\lceil \frac{m_1 + \cdots + m_{s-1} + j - 1}{j} \right\rceil$, we get

$$t-1 \geq \left\lceil \frac{(m_1 - 1) + \cdots + (m_j - 1) + m_{j+1} + \cdots + m_{s-1} + j - 1}{j} \right\rceil.$$

Consider

$$Z_1 = (m_1 - 1)P_1 + \cdots + (m_j - 1)P_j + m_{j+1}P_{j+1} + \cdots + m_{s-1}P_{s-1} + m_sP_s.$$

By the inductive assumption we can find $(t-1)$ hyperplanes, say L_2, \dots, L_t , avoiding P_s such that

$$L_2 \cdots L_t \in \wp_1^{m_1-1} \cap \cdots \cap \wp_j^{m_j-1} \cap \wp_{j+1}^{m_{j+1}} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

Moreover, since $L_1 \in \wp_1 \cap \cdots \cap \wp_j$, we get

$$L_1 L_2 \cdots L_t \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

□

We can compute the regularity index of fat points whose support is in Rnc-j.

Proposition 4.4. *Let $Z = m_1 P_1 + \cdots + m_s P_s$ be a set of fat points in \mathbb{P}^n . If P_1, \dots, P_s are in Rnc-t, then*

$$\text{reg}(Z) = \max\{D_j | j = 1, \dots, t\},$$

where

$$D_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ are in Rnc-j} \right\}.$$

Proof. We may assume that $m_1 \geq \cdots \geq m_s$. We will argue by induction on s . If $s = 1$, then $\text{reg}(Z) = m_1 - 1 = D_1$. If $s \geq 2$, we consider two following cases:

Case $t = 1$: Then P_1, \dots, P_s lie on a line and $D_1 = m_1 + \cdots + m_s - 1 = \max\{D_j | j = 1, \dots, n\}$. By Proposition 4.1 we have $\text{reg}(Z) = D_1$.

Case $t \geq 2$: Since P_1, \dots, P_s are in Rnc-t, there is the least integer $p \leq t$ such that P_1, \dots, P_s are in Rnc-p. Then

$$\begin{aligned} D_1 &= m_1 + m_2 - 1 \geq D_2 \geq \cdots \geq D_{p-1}, \\ D_p &= \left\lfloor \frac{m_1 + \cdots + m_s + p - 2}{p} \right\rfloor \geq D_{p+1} \geq \cdots \geq D_n. \end{aligned}$$

So, $\max\{D_j | j = 1, \dots, n\} = \max\{D_j | j = 1, \dots, t\} = \max\{D_1, D_p\}$. Hence, by Theorem 3.3 we get

$$\text{reg}(Z) \geq \max\{D_1, D_p\}.$$

It suffices to prove that

$$\text{reg}(Z) \leq \max\{D_1, D_p\}.$$

Put $Z_1 = m_1 P_1 + \cdots + m_{s-1} P_{s-1}$ and $J = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$. We have $\text{reg}(Z_1) = \text{reg}(R/J)$. By inductive hypothesis we have

$$\text{reg}(Z_1) = \max\{D'_j | j = 1, \dots, t\},$$

where

$$D'_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \in \{P_1, \dots, P_{s-1}\} \text{ and are in Rnc-j} \right\}.$$

Since $\{P_1, \dots, P_{s-1}\} \subset \{P_1, \dots, P_{s-1}, P_s\}$, we have $D'_j \leq D_j$ for $j = 1, \dots, t$. Therefore, we get

$$\text{reg}(R/J) \leq \max\{D_1, D_p\}. \quad (3)$$

Consider $R/(J + \wp_s^{m_s})$. We may assume that $P_s = (1, 0, \dots, 0)$, $P_1 = (0, \underbrace{1}_2, 0, \dots, 0)$, \dots , $P_p = (0, \dots, 0, \underbrace{1}_{p+1}, 0, \dots, 0)$. For every monomial $M = X_1^{c_1} \cdots X_n^{c_n}$, $c_1 + \dots + c_n = i$, $i = 0, \dots, m_s - 1$. Put

$$m'_l = \begin{cases} m_l - i + c_l & \text{for } l = 1, \dots, p, \\ m_l & \text{for } l = p+1, \dots, s-1. \end{cases}$$

Put $J' = \wp_1^{m'_1} \cap \dots \cap \wp_{s-1}^{m'_{s-1}}$. By Proposition 2.4 we can find

$$t = \max \left\{ m'_l, \left\lceil \frac{m'_1 + \dots + m'_{s-1} + p - 1}{p} \right\rceil \mid l = 1, \dots, s-1 \right\}$$

hyperplanes, say H_1, \dots, H_t , avoiding P_s such that

$$H_1 \cdots H_t \in J'.$$

Since $M \in \wp_1^{i-c_1} \cap \dots \cap \wp_p^{i-c_p}$ and $J' = \wp_1^{m'_1-i+c_1} \cap \dots \cap \wp_p^{m'_p-i+c_p} \cap \wp_{p+1}^{m'_{p+1}} \cdots \cap \wp_{s-1}^{m'_{s-1}}$, we get

$$H_1 \cdots H_t M \in J.$$

By Lemma 2.3 we get

$$\text{reg}(R/(J + \wp_s^{m_s})) \leq \max\{t + i \mid i = 1, \dots, m_s - 1\} \leq \max\{D_1, D_p\}. \quad (4)$$

Put $I = J \cap \wp_s^{m_s}$. We have $\text{reg}(Z) = \text{reg}(R/I)$. From (3), (4) and Lemma 2.1 we have

$$\text{reg}(Z) \leq \max\{D_1, D_p\}.$$

□

In Proposition 4.4, if P_1, \dots, P_s are in Rnc-n in \mathbb{P}^n , then we get Proposition 7 in [4].

References

- [1] E. Ballico, O. Dumitrescu, and E. Postinghel, *On Segre's Bound for fat points in P^n* , J. Pure and Appl. Algebra **220**, Issue 6, (2016), 2307-2323.
- [2] E. Ballico, *On the Segre upper bound of the regularity for fat points in \mathbb{P}^4 , II*, IJPAM, Vol **102**, No.2, (2015), 301-347.
- [3] B. Benedetti, G. Fatabbi and A. Lorenzini, *Segre's Bound and the case of $n+2$ fat points of P^n* , Comm. Algebra **40** (2012), 395-403.

- [4] M.V. Catalisano, N.V. Trung and G. Valla, *A sharp bound for the regularity index of fat points in general position*, Proc. Amer. Math. Soc. **118** (1993), 717-724.
- [5] E.D. Davis and A.V. Geramita, *The Hilbert function of a special class of 1-dimensional Cohen-Macaulay graded algebras*, The Curves Seminar at Queen's, Queen's Papers in Pure and Appl. Math. **67** (1984), 1-29.
- [6] G. Fatabbi, *Regularity index of fat points in the projective plane*, J. Algebra **170** (1994), 916-928.
- [7] G. Fatabbi, A. Lorenzini *On a sharp bound for the regularity index of any set of fat points*, J. Pure and Appl. Algebra **161** (2001), 91-111.
- [8] W. Fulton, *Algebraic Curves*, Math. Lect. Note Series, Benjamin 1969.
- [9] B. Segre, *Alcune questioni su insiemi finiti di punti in geometria algebrica*, Atti. Convegno. Intern. di Torino 1961, 15-33.
- [10] P.V. Thien, *Segre bound for the regularity index of fat points in \mathbb{P}^3* , J. Pure and Appl. Algebra **151** (2000), 197-214.
- [11] P.V. Thien, *Sharp upper bound for the regularity of zero-schemes of double points in \mathbb{P}^4* , Comm. Algebra **30** (2002), 5825-5847.
- [12] P.V. Thien, *Regularity index of $s + 2$ fat points not on a linear $(s - 1)$ -space*, Comm. Algebra **40** (2012), 3704-3715.

Phan Van Thien,
Department of Mathematics, Hue Normal University,
Vietnam
E-mail: tphanvannl@yahoo.com